

Polymer Effects on Heat Transport in Laminar Boundary Layer Flow

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We consider a laminar Prandtl-Blasius boundary layer flow above a slightly heated horizontal plate and study the effect of polymer additives on the heat transport. We show that the action of the polymers can be understood as a space-dependent effective viscosity that first increases from the zero-shear value at the plate to a maximum then decreases exponentially back to the zero-shear value far away from the plate. We find that with such an effective viscosity, both the horizontal and vertical velocities near the plate are decreased thus leading to an increase in the friction drag and a decrease in the heat transport in the flow.

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It has been known for more than 60 years that adding polymers into turbulent wall-bounded flows can reduce the friction drag significantly (see, for example, [1, 2] and references therein). A reduction in friction drag is equivalent to an enhancement in mass transport. Thus the effect of polymer additives on mass transport has been studied extensively during the past 60 or so years. On the other hand, the effect of polymers additives on heat transport is much less studied. Recently, an experimental study reported [3] that adding polymers to turbulent Rayleigh-Bénard (RB) convection of water, confined within a cylindrical cell heated from below and cooled on top, reduces the heat transport. In turbulent RB convection, there is an exact balance [4] between the heat transport and the energy and thermal dissipation rates. Contributions to the energy and thermal dissipation rates come from the bulk of the flow as well as the boundary layers [5]. For moderate thermal forcing as in the above-mentioned experimental study [using water at a moderate Rayleigh number (Ra) of 10^{10}], contributions from the boundary layers are dominant or significant [5]. Furthermore, there has been experimental evidence [6] that the average velocity and temperature boundary layer profiles in turbulent RB convection at moderate Ra could be described by the profiles in steady-state Prandtl-Blasius boundary layer flow [7, 8]. These experimental observations thus suggest the possibility of understanding the polymer effects on heat transport in turbulent RB convection at moderate Ra by studying the effect of polymer additives on heat transport in laminar Prandtl-Blasius boundary layer flow.

We have carried out such a study. In this Letter, we report our work and discuss our results. Physically, we can think of this boundary layer flow as the flow near the bottom plate of the convection cell in turbulent RB convection.

For the Prandtl-Blasius boundary layer flow, the equation of motion for the velocity field $v_x\hat{x} + v_y\hat{y}$ is:

$$v_x\partial_x v_x + v_y\partial_y v_x = \nu\partial_{yy}^2 v_x \quad (1)$$

Here x denotes the direction along the plate, y denotes the direction away from the plate, and ν is the kinematic viscosity of the fluid. The crucial point about Eq. (1) is that the viscous term is balanced by the nonlinear advection term and that $\partial_y \gg \partial_x$. The latter is satisfied for flows with large Reynolds number (Re). Introducing the stream function

$$\Psi(x, y) \equiv \sqrt{\nu x U} \phi(\xi) \quad \text{with} \quad \xi \equiv \sqrt{\frac{U}{\nu x}} y \quad (2)$$

we obtain the famous Blasius equation

$$2\phi_{\xi\xi\xi} + \phi\phi_{\xi\xi} = 0 \quad (3)$$

Here ϕ_{ξ} denotes $\partial_{\xi}\phi$. The boundary conditions are $v_x = v_y = 0$ at the plate and $v_x \rightarrow U$ far away from the plate, leading to:

$$\phi(0) = \phi_{\xi}(0) = 0 ; \quad \phi_{\xi}(\infty) = 1 \quad (4)$$

Writing the temperature field as

$$T(x, y) = T_0 + (T_1 - T_0)\theta(\xi) \quad (5)$$

where T_0 and T_1 are respectively the temperatures at the plate and far away from the plate (or at the center of the convection cell in the case of turbulent RB convection). Then θ satisfies the equation

$$2\theta_{\xi\xi} + \text{Pr} \phi\theta_{\xi} = 0 \quad (6)$$

with the boundary conditions:

$$\theta(0) = 1 ; \quad \theta(\infty) = 0 \quad (7)$$

and $\text{Pr} = \nu/\kappa$ is the Prandtl number.

Here, we want to investigate the effect of polymers on the heat transport in this Prandtl-Blasius flow. The polymers produces an additional stress in the momentum equation of the fluid. This polymer stress depends on the amount of stretching of the polymers and is thus a

function of the dimensionless conformation tensor R_{ij} of the polymers. Let the vector \vec{d} denote the polymer end-to-end distance and ρ_0 be the polymer radius in the unstretched regime, then R_{ij} is the average over N (where $N \gg 1$) polymers in a small region around the point (x, y) of the product $d_i d_j / \rho_0^2$, i.e., $R_{ij} = N^{-1} \Sigma d_i d_j / \rho_0^2$. In the simplest Oldroyd-B model of polymers [9], the polymer stress is given by $(\nu_p / \tau)(R_{ij} - \delta_{ij})$, where τ is the (longest) relaxation time of the polymers, and ν_p is the polymer contribution to the viscosity of the solution at zero shear, which depends on the concentration of the polymers. Thus in the presence of the polymers, the equation of motion for the velocity field is modified by an additional stress that depends on R_{ij} . By employing the same ideas leading to the Prandtl-Blasius equation (1), we have

$$v_x \partial_x v_x + v_y \partial_y v_x = \nu \partial_{yy}^2 v_x + \frac{\nu_p}{\tau} \partial_y R_{xy} \quad (8)$$

Applying the transformation by ξ to Eq. (8) does not generally lead to a similarity solution in that explicit appearance of x remains in the equation for ϕ . This is known in the literature. Similarity solution have been obtained in some special cases with the streamwise velocity and temperature going to some specific x -dependent functions when far away from the plate [10, 11]. These velocity and temperature boundary conditions do not, however, have direct physical relevance. Here to circumvent this difficulty, we recall that the Prandtl-Blasius flow is meant to be applicable when x is large (such that $\partial_y \gg \partial_x$). Thus we make the following approximations by putting $x = L$ where $L \gg 1$ is the length of the plate:

$$v_y \approx \frac{1}{2} \sqrt{\frac{\nu_0 U}{L}} (\xi \phi_\xi - \phi) \quad (9)$$

$$\partial_x \approx -\frac{\xi}{2L} \frac{d}{d\xi}; \quad \partial_y \approx \sqrt{\frac{U}{\nu_0 L}} \frac{d}{d\xi} \quad (10)$$

and we have also replaced ν in ξ by $\nu_0 = \nu + \nu_p$, the zero-shear viscosity of the polymer-laden fluid. That is, in the presence of polymers, we have

$$\xi = \sqrt{\frac{U}{\nu_0 x}} y \quad \text{with polymers} \quad (11)$$

With these approximations, the transformation of ξ leads to a similarity solution. The resulting modified Prandtl-Blasius equation is:

$$-\frac{1}{2} \phi \phi_{\xi\xi} = (1 - \gamma) \phi_{\xi\xi\xi} + \frac{\gamma}{\text{Wi}\sqrt{\text{Re}}} \frac{d}{d\xi} R_{xy} \quad (12)$$

where the Weissenberg number (Wi) and the Reynolds number (Re) are defined as

$$\text{Wi} \equiv \frac{\tau U}{L}, \quad \text{Re} \equiv \frac{UL}{\nu_0} \quad (13)$$

and $\gamma \equiv \nu_p / \nu_0$ is a function of the polymer concentration. As usual in the Prandtl-Blasius approximation, all terms of the order of $1/\text{Re}$ are neglected in Eq. (12).

We are interested in studying whether and how the heat transport is affected by the polymers. In turbulent RB convection, it is common to measure the heat flux Q in terms of the dimensionless Nusselt number (Nu), which is the ratio of Q to the heat flux when there is only conduction and is defined by

$$\text{Nu} = \frac{Q}{2k(T_1 - T_0)/H} = \frac{\langle -\frac{\partial T}{\partial y} \Big|_{y=0} \rangle_A}{2(T_1 - T_0)/H} \quad (14)$$

Here k is the thermal conductivity of the fluid, H is the height of the convection cell, and $\langle \dots \rangle_A$ is the average over the cross section of the cell. For the Prandtl-Blasius flow, taking $H = L$ and dropping the numerical factor, Nu can be estimated as

$$\text{Nu} = \sqrt{\frac{UL}{\nu_0}} [-\theta_\xi(0)] \quad (15)$$

We want to state our major result: we will show that the action of the polymers is to give rise to a space-dependent effective viscosity that increases from the zero-shear value at the boundary. This increase of viscosity leads to an enhancement in drag, which in turn induces a reduction in heat flux. To solve Eq. (12), one needs to supplement it with specific information of R_{xy} . In a fluid flow of velocity \vec{v} , the dimensionless polymer end-to-end distance $\vec{l} = \vec{d}/\rho_0$ obeys the differential equations

$$\frac{dl_i}{dt} = -\frac{1}{2\tau}(l_i - l_{0i}) + l_j \partial_j v_i + \text{thermal noise} \quad (16)$$

where $l_{0x} = \cos \alpha$, $l_{0y} = \sin \alpha$ and α is a random angle uniformly distributed in $[0, 2\pi]$. For the two-dimensional Prandtl-Blasius flow and neglecting thermal noise, we can rewrite Eq. (16) as

$$\frac{L}{U} \frac{dl_x}{dt} = -\frac{(l_x - l_{0x})}{2\text{Wi}} - \frac{1}{2} \xi \phi_{\xi\xi} l_x + \sqrt{\text{Re}} \phi_{\xi\xi} l_y \quad (17)$$

$$\frac{L}{U} \frac{dl_y}{dt} = -\frac{(l_y - l_{0y})}{2\text{Wi}} - \frac{1}{4\sqrt{\text{Re}}} \xi^2 \phi_{\xi\xi} l_x + \frac{1}{2} \xi \phi_{\xi\xi} l_y \quad (18)$$

In order to obtain l_i as a function of ξ , we assume that each polymer follows the streamline of the flow such that $d\xi/dt \approx -(U/2L)\phi$.

The quantity R_{xy} is the average of $l_x l_y$ over all the polymers contained in a small volume centered near the point (x, y) and over the angle α . Equations (12), (17) and (18) are to be solved consistently. The procedure of how these equations are solved is rather involved and the details will be discussed in a forthcoming paper [12]. Here we show that one can already obtain useful physical insights by assuming a balance, for any ξ , between the stretching of the polymers due to the velocity field and the relaxation, represented by the term with the factor $1/\text{Wi}$. This is equivalent to say that, in the small-Wi limit, we can neglect the LHS of Eqs. (17) and (18). Neglecting terms $O(1/\sqrt{\text{Re}})$ and averaging over α , we obtain, to the order in Wi^2 :

$$R_{xy} = (1 + \xi \phi_{\xi\xi} \text{Wi}) \phi_{\xi\xi} \text{Wi} \sqrt{\text{Re}} \quad (19)$$

Substituting Eq.(19) in Eq. (12), we have

$$2\phi_{\xi\xi\xi} + \phi\phi_{\xi\xi} + 2\partial_{\xi}[f(\xi)\phi_{\xi\xi}] = 0 \quad (20)$$

where $f(\xi) \equiv A\xi\phi_{\xi\xi}$ and $A = \text{Wi}\gamma$ is a function of Wi and the polymer concentration. Equation (20) demonstrates that the effect of the polymers is equivalent to introducing a space dependent viscosity. To see this, go back to Eq. (8) and assume that $\nu_p R_{xy}/\tau$ is equivalent to the product of a y -dependent effective viscosity, $\nu_p + \nu_0 f(y)$, and the velocity gradient, $\partial_y v_x$, we obtain:

$$v_x \partial_x v_x + v_y \partial_y v_x = \nu \partial_y^2 v_x + \partial_y \{[\nu_p + \nu_0 f(y)] \partial_y v_x\} \quad (21)$$

which is exactly equivalent to Eq. (20) once the rescaling in the variable ξ is performed.

We solve Eq. (20) for $A = 1$, and show $f(\xi)$ in Fig. 1. It can be seen that $f(\xi)$ increases from zero at the boundary and then decreases almost exponentially towards zero. This increase of the effective viscosity near the plate allows us to predict the effect of the polymer on the flow. First, we expect an increase in the effective viscosity near the plate would give rise to an increase of the drag. We measure the drag by the drag coefficient:

$$C = \frac{\nu_0 \partial_y v_x|_{y=0}}{U^2/2} = \frac{1}{2\sqrt{\text{Re}}} \phi_{\xi\xi}(0) \quad (22)$$

and compare C to the value C_0 for the flow without polymers. We expect C/C_0 would be greater than 1 for all $A > 0$. Such an enhancement in drag would in turn result in a reduction of the mass throughput in the horizontal direction and thus a decrease in the horizontal velocity near the boundary. These effects are indeed observed. In Fig. 1 we show the horizontal velocity v_x^0 and v_x^p respectively for the case without polymers (denoted by the superscript 0) and with polymers (denoted by the superscript p), and also compare the vertical velocities, v_y^p and v_y^0 , with and without polymers. It can be seen that in the presence of polymers, both the horizontal and vertical velocities decrease in the region near the plate.

Next, we show that drag enhancement implies a reduction of the heat transport or Nu . Upon double integration of Eq. (6) by ξ , we obtain:

$$-\theta_{\xi}(0) = \left\{ \int_0^{\infty} ds_1 \exp[-\text{Pr} \int_0^{s_1} \phi(s_2) ds_2 / 2] \right\}^{-1} \quad (23)$$

which tells us that Nu is a functional of Φ , where $\Phi(\xi) \equiv \int_0^{\xi} \phi(s) ds$. We can thus calculate δNu , the variation in Nu induced by a variation in Φ , $\delta\Phi(\xi)$:

$$\delta\text{Nu} = \frac{\text{Pr} \text{Nu}^2}{2} \int_0^{\infty} \exp[-\text{Pr} \Phi(s)/2] \delta\Phi(s) ds \quad (24)$$

Since $v_x = U\phi_{\xi}$, the mass throughput in the x direction across a distance ξ is given by $U\phi(\xi)$, and thus a reduction in mass throughput implies an $\delta\Phi < 0$. Therefore, Eq. (24) shows that drag enhancement implies a reduction in Nu or the heat transport.

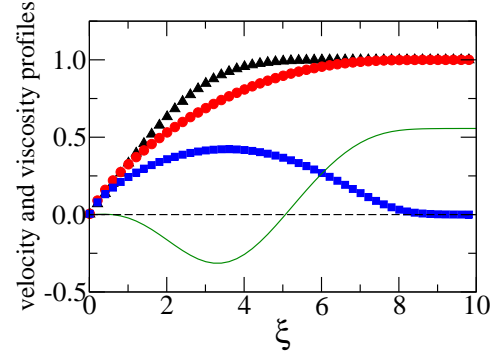


FIG. 1: Viscosity and velocity profiles obtained by solving Eq. (20) with $A = 0$ and $A = 1$. We show the space-dependent viscosity $f(\xi)$ for $A = 1$ (squares), the horizontal velocity, $v_x^0(\xi)/U$ (triangles) and $v_x^p(\xi)/U$ (circles), without ($A = 0$) and with polymers ($A = 1$), and the difference in vertical velocity, $[v_y^p(\xi) - v_y^0(\xi)]/(U/2\sqrt{\text{Re}})$ (solid line).

The above considerations are not restricted to the particular form of $f(\xi) = A\xi\phi_{\xi\xi}$ obtained in the small- Wi limit but are true for a generic form of $f(\xi)$ which displays the same features of first increasing near the boundary to some maximum then decreasing to zero far away from the boundary as shown in Fig. 1. In particular, we show that this is the case by using a different form of $f(\xi)$:

$$f(\xi) = A\xi \exp(-\xi) \quad (25)$$

that has similar qualitative features. The constant A relates the polymer characteristics to the conformation tensor R_{ij} , and generally increases with Wi and the concentration of the polymers.

Solving Eq. (25) with Eq. (20), we again find an increase in drag and a reduction in heat transport. Moreover as shown in Fig. 2, the extent of the effect is increased by increasing A , i.e., by increasing the effect of the polymers into the system. For the range of A shown in Fig. 2, the amount of the reduction of heat flux is quantitatively consistent with that observed in Ref. [3].

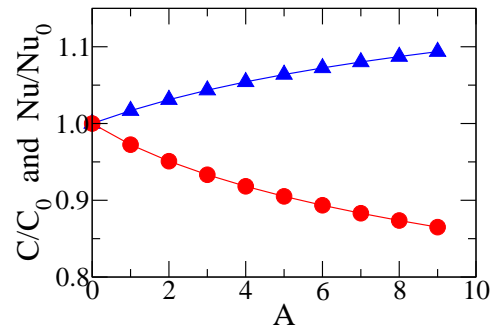


FIG. 2: Dependence of C/C_0 , (triangles) and Nu/Nu_0 , (circles) on A for $f(\xi)$ given by Eq. (25).

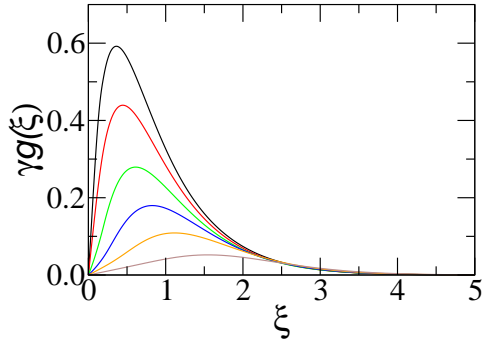


FIG. 3: $\gamma g(\xi)$ as a function of Wi at fixed $\gamma = 0.2$ obtained by the numerical solution of the full nonlinear problem, Eqs. (12), (17) and (18). Wi increases from bottom to top, taking the values 0.5, 1.0, 1.5, 2.0, 2.5, and 2.8 respectively.

In the remaining of this Letter, we show that the results that we have discussed capture the qualitative physical picture of the full nonlinear problem, defined by Eqs. (12), (17) and (18). We have solved the full problem, by integrating numerically the equations using an iterated procedure [12]. In particular, we show that $R_{xy} = [1 + g(\xi)]\phi_{\xi\xi}Wi\sqrt{Re}$ for some function $g(\xi)$. Substituting this into Eq. (12) gives

$$2\phi_{\xi\xi\xi} + \phi\phi_{\xi\xi} + 2\partial_{\xi}[\gamma g(\xi)\phi_{\xi\xi}] = 0 \quad (26)$$

Comparing Eq. (26) with Eq. (20) for the small- Wi limit, we see that $\gamma g(\xi)$ plays the role of $f(\xi)$. In Fig. 3, we show $\gamma g(\xi)$ for different values Wi at a fixed value of $\gamma = 0.2$. Notice that the functional shape of $g(\xi)$ is similar to the one used in Eq. (25). Not surprisingly, we find drag enhancement and heat transport reduction for the full nonlinear problem as shown in Fig. 4.

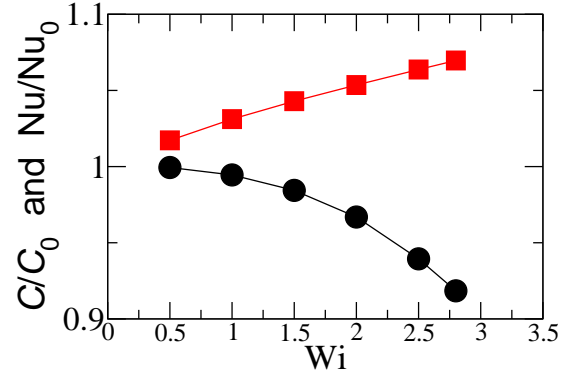


FIG. 4: Drag enhancement and heat transport reduction for the full nonlinear problem and the dependence of C/C_0 (squares) and Nu/Nu_0 (circles) on Wi .

In summary, we have shown the following results:

- (1) It is possible to write a self consistent set of equations for Prandtl-Blasius boundary layer flow with polymers.
- (2) The physical effect of the polymers is equivalent to introducing a space-dependent effective viscosity that is increasing near the boundary.
- (3) An increasing viscosity near the boundary leads to drag enhancement, which in turn induces a reduction in heat transport.

Our theory may thus explain the recent experimental results observed in turbulent RB convection [3].

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